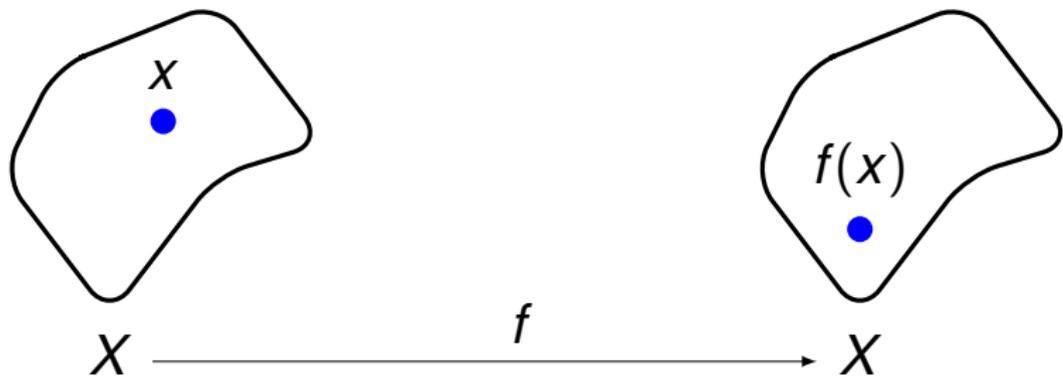# Orbit structure in CR-dynamical systems

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Topological Dynamical Systems

Definition
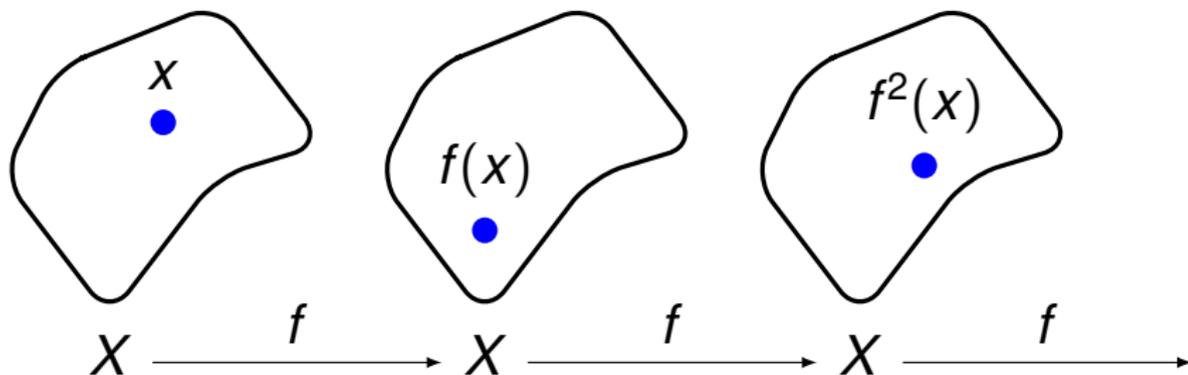
For a non-empty compact metric space *X* and continuous function $f : X \to X$, we say (X, f) is a *topological dynamical system*.

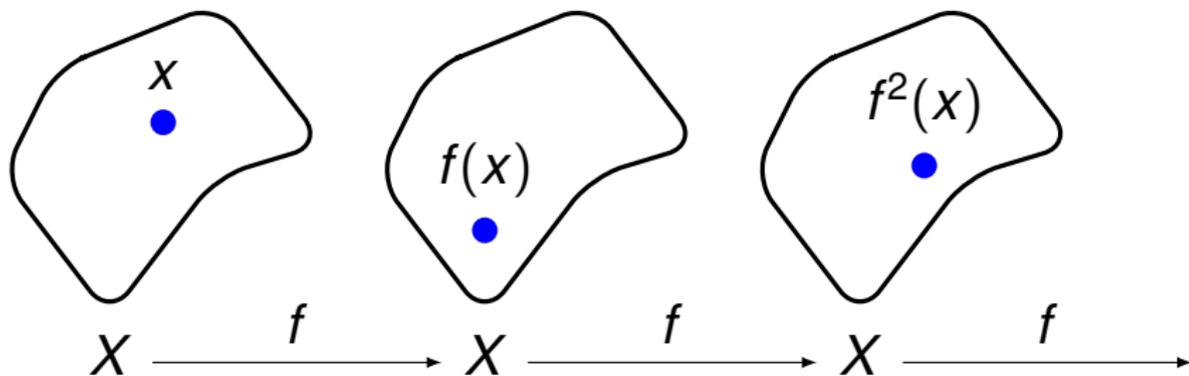# Topological Dynamical Systems

Definition

Let (X, f) be a topological dynamical system. Then the *trajectory* of a point $x \in X$ is the sequence $\langle x, f(x), f^2(x), \ldots \rangle$.

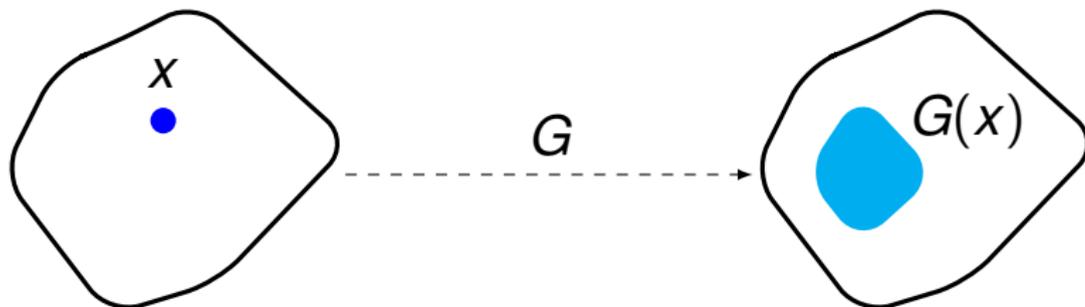# Topological Dynamical Systems

Definition

Let (X, f) be a topological dynamical system. We say $x \in X$ is a *transitive point*, if $\{x, f(x), f^2(x), \ldots\}$ is dense in X.

CR-Dynamical Systems

Definition

For a non-empty compact metric space X and non-empty closed $G \subseteq X \times X$, we say (X, G) is a *CR-Dynamical System*. For each $x \in X$, we denote by $G(x)$, the set $\{y \in X \mid (x, y) \in G\}$.
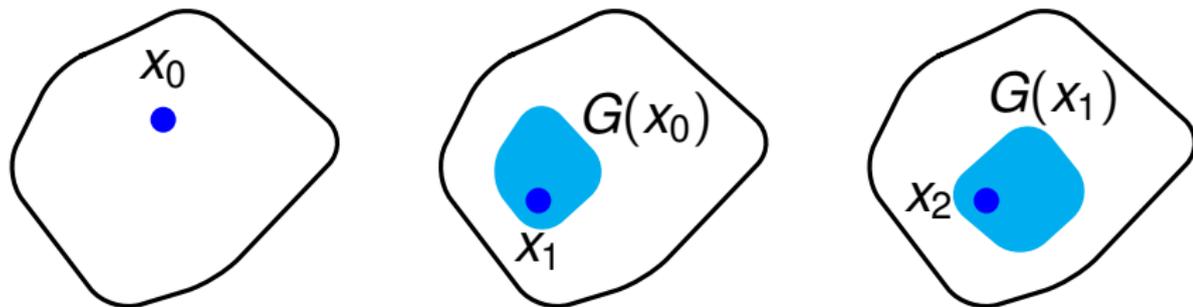
Trajectories

Definition

Let (X, G) be a CR-dynamical system. Then a *trajectory* of a point $x \in X$ is a sequence $\langle x_n \mid n \in \mathbb{N} \rangle$ such that

- $x_0 = x$; and
- $x_{n+1} \in G(x_n)$ for all $n \in \mathbb{N}$.

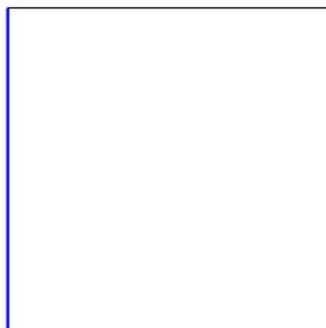Denote by $T_G^+(x)$ the set of all trajectories of x.

Legality

Definition

Suppose (X, G) is a CR-dynamical system.

- We say $x \in X$ is *legal* if $T_G^+(x) \neq \varnothing$
- We say $x \in X$ is *illegal* if $T_G^+(x) = \varnothing$.
- Denote by $\text{legal}\,(G)$ the set of legal points.

$$X = [0, 1]$$

$$G = \{0, 1\} \times X$$

$$\text{legal}\,(G) = \{0, 1\}$$
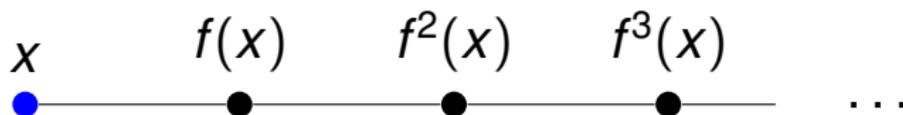
Orbit structure in classical dynamics

Figure: Orbit structure in topological dynamical systems

Trees

Definition

We say a partially ordered set $(T, \leq)$ is a *rooted tree*, if there exists a unique point $r \in T$, called the *root*, such that for each $x \in T$:

- $r \leq x$.
- $(\{y \in T \mid y \leq x\}, \leq)$ is well-ordered.

r

Levels

$\text{level}_n(T)$

$\vdots$

$\text{level}_3(T)$

$\text{level}_2(T)$

$\text{level}_1(T)$ $\quad\mathcal{L}_2(T)$

$\text{level}_0(T)$

r

Branches

Definition

Let $(T, \leq)$ be a tree. We call a maximally well-ordered subset of T a *branch*. We denote by $\mathcal{B}_\infty(T)$, the set of infinite branches in T.
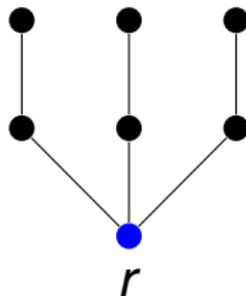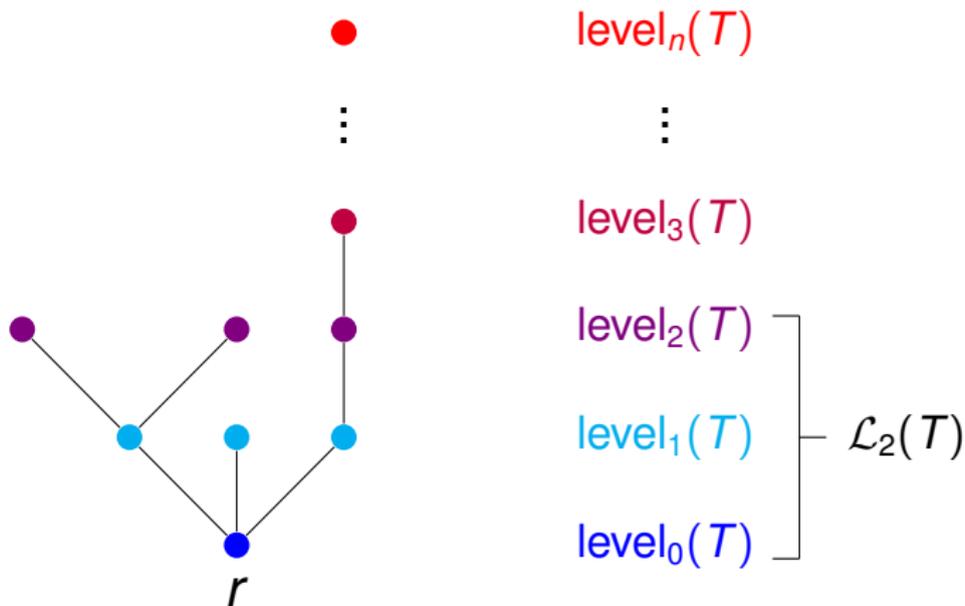
Paths in G

Definition

Let (X, G) be a CR-dynamical system, $x, y \in X$ and $n \in \mathbb{N}$. We say $\gamma = x_0 \ldots x_n$ *is an x-path in G*, if

- $x_0 = x$; and

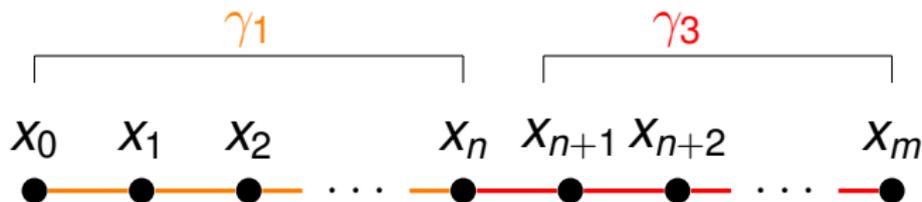- $(x_k, x_{k+1}) \in G$ for each $k \in \{0, 1, \ldots, n-1\}$.

We denote by x_γ the endpoint of our path γ, i.e., $x_\gamma = x_n$. If in addition $x_\gamma = y$, we say $\gamma = x_0 \ldots x_n$ *is a path from x to y in G*. We denote by $\Gamma_G(x)$, the set of x-paths in G.

x_0 x_1 x_2 x_n

$\bullet \quad\quad \bullet \quad\quad \bullet \quad \cdots \quad \bullet$

Extension of paths in G

Definition

Let (X, G) be a CR-dynamical system and $x \in X$. Let $\gamma_1, \gamma_2 \in \Gamma_G(x)$. We say γ_1 *extends* to γ_2, if $\gamma_2 = \gamma_1 \gamma_3$ for some path γ_3 in G.

$$\gamma_1 \qquad \gamma_3$$

$$x_0 \quad x_1 \quad x_2 \quad \cdots \quad x_n \quad x_{n+1} \quad x_{n+2} \quad \cdots \quad x_m$$

Extension of paths in *G*

Definition

Let (X, G) be a CR-dynamical system and $x \in X$. Let $\gamma_1, \gamma_2 \in \Gamma_G(x)$. We say γ_1 *extends* to γ_2, if $\gamma_2 = \gamma_1 \gamma_3$ for some path γ_3 in *G*.

Theorem

Let (X, G) be a CR-dynamical system and $x \in X$. Then, $(\Gamma_G(x), \leq)$ is a tree, where

$$\gamma_1 < \gamma_2 \text{ if, and only if, } \gamma_1 \text{ extends to } \gamma_2,$$
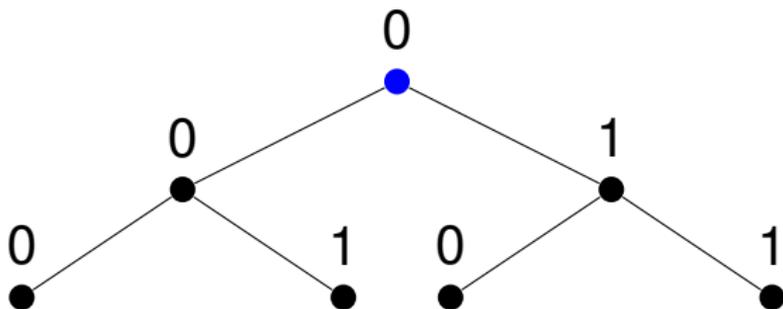
for each $\gamma_1, \gamma_2 \in \Gamma_G(x)$.

Transitivity trees

Definition

Let (X, G) be a CR-dynamical system. The *Transitivity tree of (X, G) with respect to x*, is the tree $(T_G(x), \leq)$, where $T_G(x) = \Gamma_G(x)$ and $\leq$ is the path extension order. If $S \subseteq T_G(x)$, we denote by S^* the set $\{x_\gamma \in X \mid \gamma \in S\}$.

Example

Let $X = \{0, 1\}$ and $G = X \times X$. Then, below we show $\mathcal{L}_2(T_G(0))$.

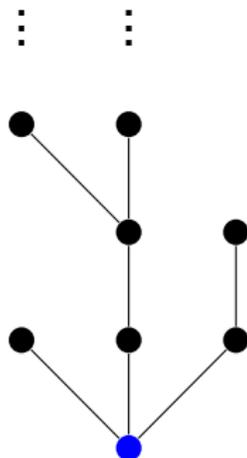We note $T_G(0)$ consists of all binary strings starting at 0, and $T_G(1)$ all binary strings starting at 1.

Figure: Transitivity tree of a legal point

Application to transitive points

Definition

Let (X, G) be a CR-dynamical system. We say $x \in \mathrm{legal}\,(G)$ is

- 1-*transitive*, if for every infinite branch $B \in \mathcal{B}_\infty(T_G(x))$ we have $\overline{B^*} = X$.
- 2-*transitive*, if there exists an infinite branch $B \in \mathcal{B}_\infty(T_G(x))$ such that $\overline{B^*} = X$.
- 3-*transitive*, if $\bigcup_{B \in \mathcal{B}_\infty(T_G(x))} B^*$ is dense in X.

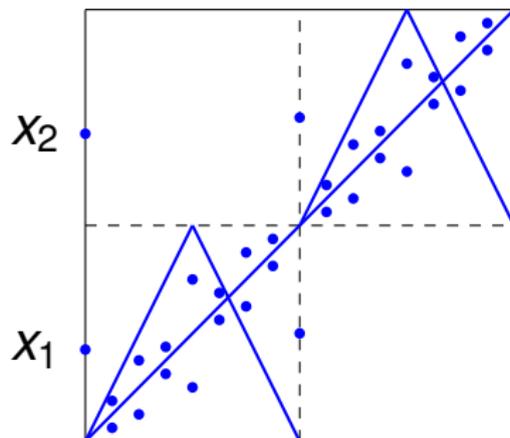$$\mathrm{trans}_1(G) \subseteq \mathrm{trans}_2(G) \subseteq \mathrm{trans}_3(G)$$

Application to transitive points

Theorem

Let (X, G) be a CR-dynamical system such that isolated $(X) \neq \varnothing$ *and* trans$_2$ $(G) \neq \varnothing$. *Then, there exists $x \in X$ such that*

$$T_{G^{-1}}(x)^* = \text{trans}_2\,(G) = \text{trans}_3\,(G).$$

Stormy Campsite Map

Thank you for your attention!